Recap & Setup

- R $\mathbb{Z}$-graded comm. noeth. ring (or: comm. graded ...)

  can assume: $R = R_{even}$ or $2R = 0$.

  Homog. spec $\mathrm{Spec}^*(R)$.

  If M is in $D^b(R)$, or just a module:

  support variety is $\mathrm{Supp}_R(M) := \{ \mathfrak{p} \mid M_\mathfrak{p} \neq 0 \}$ (in $D^b(R)$, i.e.: not acyclic)

  $$= \{ \mathfrak{p} \mid H(M)_\mathfrak{p} \neq 0 \} = V(\mathrm{ann}_R(HM))$$

  $(-)_\mathfrak{p}$ is exact; if HM is f.g.

- G finite group, k field, $\mathrm{char}(k) = p \mid |G|$.

  Recall $H^*(G,k)$: gr. comm. f.g. k-alg.

  not comm, so we can take: $H^\bullet(G) := \begin{cases} H^{even}(G,k) & p \text{ odd} \\ H^*(G,k) & p = 2. \end{cases}$

  $M \in D^b(kG)$, $\mathrm{Ext}^*_{kG}(M,M)$ is a f.g. $H^*(G)$-module, so:

  $V_G(M) := \mathrm{Supp}_{H^\bullet(G)} \mathrm{Ext}^\bullet_{kG}(M,M)$.

  More generally, "relative supp. varieties":

  $V_G(M,N) := \mathrm{Supp}_{H^\bullet(G)} \mathrm{Ext}^*_G(M,N)$.

  We write: $V_G := V_G(k) = \mathrm{Spec}^* H^\bullet(G)$

- Note that: $V_G(M) \underset{f.g.}{=} V(\mathrm{ann}_{H(G)} \mathrm{Ext}^\bullet_G(M,M))$

Obs: $\mathrm{ann}_{HG} \mathrm{Ext}^\bullet(M,M) = \bigcup_{\text{any } N} \mathrm{ann}\, \mathrm{Ext}^\bullet(N,M) \cup \mathrm{ann}\, \mathrm{Ext}^\bullet(M,N) =$

$$= \{ \zeta \in H^\bullet(G) \mid \zeta \cdot \mathrm{id}_M = 0 \},$$

because $\mathrm{Ext}^\bullet(M,M)$ is an $H^\bullet(G)$-algebra!

<u>Stable setting</u>:

If M is in kG-<u>mod</u>, $H^\bullet(G) \subseteq \mathrm{Ext}^\bullet_{kG}(k,k) \subseteq \underbrace{\widehat{\mathrm{Ext}}^\bullet(k,k)}_{\text{Tate cohomology}}$

$$\widehat{\mathrm{Ext}}^\bullet(k,k) = \underline{\mathrm{End}}^\bullet(k,k)$$

$\rightsquigarrow$ $H^\bullet(G) \curvearrowright \widehat{\mathrm{Ext}}^\bullet_G(M,M) := \underline{\mathrm{Hom}}^\bullet(M,M)$

Define: $\underline{V}_G(M) := \{ \mathfrak{p} \neq H^+(G) \mid \underbrace{\widehat{\mathrm{Ext}}^\bullet(M,M)}_{\text{not f.g. anymore, but:}}{}_{\mathfrak{p}} \neq 0 \} \subseteq \mathrm{proj}\, H^\bullet(G)$.

<u>Note</u>: $\widehat{\mathrm{Ext}}(M,M)_{\mathfrak{p}} = 0 \iff \widehat{\mathrm{Ext}}^{\geq 0}(M,M)_{\mathfrak{p}} = 0$

and $\mathrm{ann}\, \widehat{\mathrm{Ext}}^\bullet(M,M) \hookleftarrow \mathrm{ann}\, \widehat{\mathrm{Ext}}^{\geq 0}(M,M)$.

Thus:

$$\underline{V}_G(M) = V(\mathrm{ann}\, \widehat{\mathrm{Ext}}^\bullet(M,M)) \subseteq \mathrm{Proj}\, H^\bullet(G) = \underline{V}_G(k).$$

<u>Note</u>:

$$\mathrm{ann} = \{ \zeta \in H^+(G) \mid \zeta \cdot \mathrm{id}_M = 0 \} \text{ in } \widehat{\mathrm{Ext}}(M,M)$$

$$= \{ \zeta \in \underbrace{H^+(G)}_{=: \mathfrak{m}} \mid \zeta \cdot \mathrm{id}_M = 0 \} \text{ in } \mathrm{Ext}(M,M)$$

$$= \mathfrak{m} \cap \mathrm{ann}_{H^\bullet(G)} \mathrm{Ext}^\bullet_{kG}(M,M).$$

It follows that $\underline{V}_G(M) = V_G(M) \setminus \{0\}$ $\quad H^+(G) = R_{\geq 1}$.

Properties

① $V_G(M)=\{0\} \iff M$ is projective

Sketch: if $V_G(M)=0$, $V_G(S,M)=0$: S a simple kG-mod:

$$\mathrm{Ext}(S,M)\otimes \mathrm{Ext}(M,M) \xrightarrow{\text{surj}} \mathrm{Ext}(S,M)$$

Now, $\sqrt{\mathrm{ann}\,\mathrm{Ext}^{\cdot}_{HG}(S,M)} = m$,

$\rightarrow$ $m^n \mathrm{ann}_{kG}\,\mathrm{Ext}^{\cdot}(S,M)=0$ for some n $\rightarrow$ $\mathrm{Ext}^{>k}(S,M)=0$

$\rightarrow$ the proj. cover of S appears only fin many often in the min. proj res of M. ($\forall$ S simple, $\exists$ fin many!)

$\rightarrow$ M has a finite proj res.

$\rightarrow$ M is projective

$\square$

② $V_G(M)=V_G(\Sigma M)=V_G(\Omega M)=V_G(M^*)$

Sketch: $\mathrm{Ext}^{\cdot}_G(M,M)\cong \mathrm{Ext}_G(\Sigma M,\Sigma M)\cong \mathrm{Ext}^{\cdot}_G(M^*,M^*)$. $\square$

③ $V_G(M\oplus N)=V_G(M)\cup V_G(N)$

(distr. $\oplus$ from Ext ...), more generally:

$M_1\rightarrow M_2\rightarrow M_3\rightarrow$ a Δ in $D^b(kG)$

then from the LES in cohomology we get:

$V_G(M_2)\subseteq V_G(M_1)\cup V_G(M_3)$. (use $(-)_{p}$ exact!).

④ $H\leq G$ subgroup $\rightsquigarrow$ get a map of graded k-algebras:

$\mathrm{res}_{G,H}: H^*(G)\rightarrow H^*(H)$ $\rightsquigarrow$ $\mathrm{res}^*_{G,H}: V_H \longrightarrow V_G$

Then for any M in $D^b(kG)$, have:

$$V_H(M|_H)=(\mathrm{res}^*_{G,H})^{-1}(V_G(M))$$

([illegible] for any group map $H\xrightarrow{f}G$)

Uses Quillen stratification ...

(5) From this, using $G \overset{\Delta}{\subseteq} G \times G$ diagonal, obtain from the:

$$V_G(M \otimes N) = V_G(M) \cap V_G(N)$$

(not so difficult).

(6) <u>Quillen stratification</u> / <u>Avrunin-Scott</u> : (See [Benson's Book])

We have $V_G(M) = \bigcup_{\substack{E \text{ elem.} \\ \text{ab.} \subseteq G}} \operatorname{res}^*_{kE}(V_E(M))$

Recall: E el. abelian $:\Leftrightarrow E \cong (\mathbb{Z}/q\mathbb{Z})^r$, q prime, $r \in \mathbb{N}$
(here only interesting for $q = p$).

Actually, Quillen strat. gives a more precise picture...

<u>Cor.</u> (Chouinard's Thm.)

M is projective ($\Leftrightarrow V_G(M) = \{0\}$)
$\Leftrightarrow$ all $M|_E$ are projective for all E el. ab. p-subgroups of G.

<u>Quillen</u>: The dimension of V_G = the max. dim. of V_E.

(7) Take any $0 \neq \zeta \in H^\bullet(G)$; If we represent ζ as autom. $\Omega^n k \xrightarrow{\zeta} k$, take kernel: $0 \to L_\zeta \to \Omega^n k \xrightarrow{\zeta} k \to 0$ surjective

<u>Theorem</u> (Carlson):

$V_G(L_\zeta) = V((\zeta))$ (is a hypersurface)

Cor: Using tensor prod. theorem, can realise any closed set as:

$$V_G(L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_m}) = V(\underbrace{(\zeta_1, \dots, \zeta_m)})$$

any ideal at all

(8) There is a converse to (3), due to Carlson:

If $V_G(M) = V_1 \cup V_2$ with $V_1 \cap V_2 = \{0\}$, then $M \cong M_1 \oplus M_2$ with $V_G(M_i) = V_i$.

I.e: The (proj.) supp var. of an indec. module is connected

Thick $\otimes$-ideals & support varieties

- Recall, a full ~~thick~~ subcat $\mathcal{C} \hookrightarrow D^b(kG)$ (or any Δed cat), is thick when closed under Σ, extensions (cones), summands.

Furthermore: $\mathcal{C}$ is a thick $\otimes$-ideal, when also:

if $M \in \mathcal{C}$, N anything, then $M \otimes N \in \mathcal{C}$.

$\underbrace{\text{Thick}^{\otimes}(M)}_{\text{or any subset}}$:= smallest thick $\otimes$-ideal containing M.

"M builds N" if $N \in \text{thick}^{\otimes}(M)$.

- Recall, $V \subseteq V_G$ is called specialization closed if: $\mathfrak{p} \in V$, $\mathfrak{q} \supseteq \mathfrak{p} \Rightarrow \mathfrak{q} \in V$.

Equivalently: V = a union of closed sets.

Def: Given $V \subseteq V_G$, write $D^b(kG)_V$ for the full subcat. of all M such that $V_G(M) \subseteq V$.

Given the properties we've seen, easy to see: $D^b(kG)_V$ is a thick tensor ideal subcategory!

Def: If $\mathcal{C} \subseteq D^b(kG)$ a thick $\otimes$-ideal, write $V_G(\mathcal{C})$ for

$$V_G(\mathcal{C}) := \bigcup_{M \in \mathcal{C}} V_G(M)$$

⇝ specialization closed, very easy to see from def.

The main Thm. of this workshop says that these two constr. are inverse bijections (also: lattice isomorphisms!)

$$\left\{ \begin{array}{c} \text{thick } \otimes\text{-ideals} \\ \text{in } D^b(kG) \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} V \subseteq V_G \text{ specializ.} \\ \text{closed subsets} \end{array} \right\}$$

$$\mathcal{C} \longmapsto V_G(\mathcal{C})$$

$$D^b(kG)_V \longleftarrow V$$

$$\cup\!| \qquad\qquad\qquad\qquad \cup\!|$$

$$\left\{ \begin{array}{c} \text{thick}^{\otimes}(M) \\ \text{for some} \\ \text{object } M \end{array} \right\} \xleftrightarrow{1-1} \{\text{closed subsets}\}$$

The hardest part of this is to establish:

Thm: If $V_G(M) \subseteq V_G(N)$ then $\text{thick}^{\otimes}(M) \subseteq \text{thick}^{\otimes}(N)$.

Example: K_4 Klein 4-group

$$\overset{\shortparallel}{G} := (\mathbb{Z}/2\mathbb{Z})^2 = \langle a, b \rangle, \quad \text{char } k = 2.$$

Note: $(1-a)^2 = (1-b)^2 = 0$ ← 2 commuting generators

$$kG \underset{\substack{\uparrow \\ G \text{ abelian}}}{\cong} k[a,b]/_{(1-a)^2, (1-b)^2} \cong \frac{k[x,y]}{x^2, y^2}$$

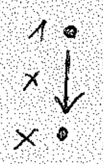
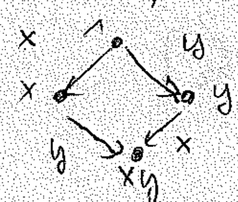
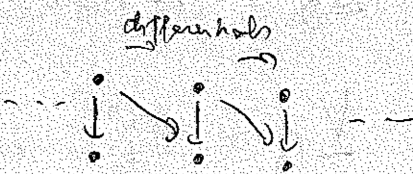
Introduce: $R = k[\mathbb{Z}/2\mathbb{Z}] = \frac{k[x]}{x^2}$

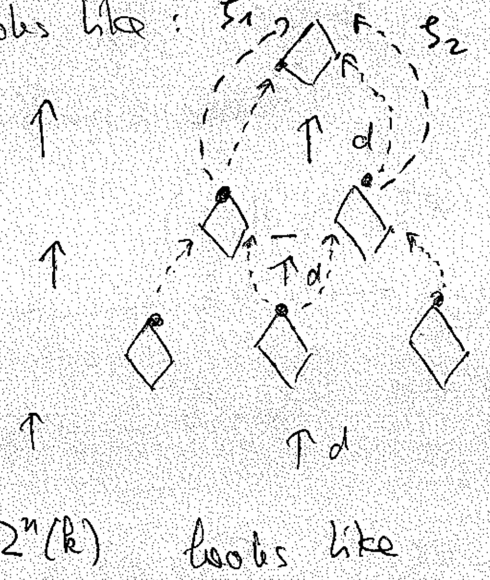
k has a minimal resolution as an R-module:

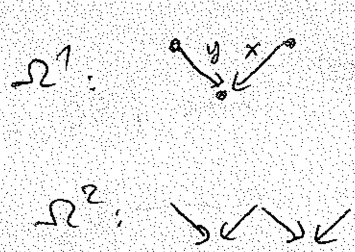
$$P_\bullet := \left(\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{\alpha} k \right)$$

By Künneth (or direct checking), $H^*(M^\cdot \otimes_k N^\cdot) \cong H^*(M^\cdot) \otimes_k H^*(N^\cdot)$

we have: $P_\bullet \otimes P_\bullet$ is a min. res. of k as a kG-module.

We draw R:  kG  Then $P_\bullet$ looks like: 

$\Rightarrow$ $P_\bullet \otimes P_\bullet$ looks like: 



In general, $\Omega^n(k)$ looks like $\bigvee\!\bigvee\!\bigvee \cdots \bigvee$ $\}\, k^{n+1}$ (top) $\}\, k^n$ (socle)

in particular: $\dim \Omega^n k = 2n+1$.

Similarly: cosyzygies $\Omega^{-n}(k) = \bigwedge\!\bigwedge\!\bigwedge \cdots \bigwedge$ dim $= n$ (top), dim $= n+1$ (socle)

As R-module: $\Omega^n_R(k) = k$, so $\mathrm{Ext}^n(k,k) = \underline{\mathrm{Hom}}(k,k) = \mathrm{Hom}(k,k) = k \ni 1$

$2 \mid |C_2|$

And one checks that $1 \overset{\cong}{\longleftrightarrow}$ the chain map:

$$\begin{tikzcd}
R \arrow[d, "x"'] \arrow[dr, equal] & R \arrow[d, "x"] \\
R \arrow[d] \arrow[dr, equal] & R \arrow[d] \\
\vdots \arrow[d] & R \\
R \arrow[r, equal] & R
\end{tikzcd}$$

From here you see $\mathrm{Ext}^\bullet_R(k,k) = k[\zeta]$

Künneth $\Rightarrow$ $H^\bullet(G) = k[\zeta_1, \zeta_2]$

Hence: $\underline{V}_G = \mathbb{P}^1_k$ $\rightsquigarrow$ this space parametrizes the indec. kG-modules (tame situation!) $\hookrightarrow$ [done by Kronecker, via the quiver $\bullet \rightrightarrows \bullet$]

$(\alpha,\beta) \in \mathbb{P}^1 \rightsquigarrow (\beta\zeta_1 - \alpha\zeta_2) \subseteq H(G)$

$$0 \longrightarrow L_\zeta = L_{(\alpha,\beta)} \longrightarrow \Omega k \longrightarrow k \longrightarrow 0$$

$$0 \longrightarrow x = \alpha\!\downarrow\downarrow\!\beta = y \longrightarrow \begin{smallmatrix} \bullet & & \bullet \\ {}_x\searrow & & \swarrow_y \\ & \bullet & \end{smallmatrix} \longrightarrow \bullet$$

$\Rightarrow V_G(L_\zeta) = V(\beta\zeta_1 - \alpha\zeta_2) = \{(\alpha,\beta)\} \subseteq \mathbb{P}^1.$

$\nwarrow$ one closed point of the projective line